# ON THE CLASSIFICATION OF q-ALGEBRAS

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*ABSTRACT.* The problem is the classification of the ideals of "free differential algebras", or the associated quotient algebras, the q-algebras; being finitely generated, unital **C**-algebras with homogeneous relations and a q-differential structure. This family of algebras includes the quantum groups, or at least those that are based on simple (super) Lie or Kac-Moody algebras. Their classification would encompass the so far incompleted classification of quantized (super) Kac-Moody algebras and of the (super) Kac-Moody algebras themselves. These can be defined as singular limits of $q$-algebras, and it is evident that to deal with the $q$-algebras in their full generality is more rational than the examination of each singular limit separately. This is not just because quantization unifies algebras and superalgebras, but also because the points "$q = 1$" and "$q = -1$" are the most singular points in parameter space. In this paper one of two major hurdles in this classification program has been overcome. Fix a set of integers $n_1, ..., n_k$, and consider the space $\mathcal{B}_Q$ of homogeneous polynomials of degree $n_1$ in the generator $e_1$, and so on. Assume that there are no constants among the polynomials of lower degree, in any one of the generators; in this case all constants in the space $\mathcal{B}_Q$ have been classified. The task that remains, the more formidable one, is to remove the stipulation that there are no constants of lower degree.

## 1. Introduction.

The classification of simple Lie algebras, achieved by Killing and Cartan near the end of the 19'th century, was a milestone in modern mathematics. Other classification problems came later: simple Lie bialgebras, super Lie algebras [BD], Kac-Moody algebras [Kc], [M], generalized Kac-Moody algebras, quantum groups. Little is known about generalized Kac-Moody algebras, and the classification of Lie superalgebras is only now nearing completion. In fact, a complete understanding of the relations that govern the latter turned out to be a quite formidable task [MZ], [GL], [B], [FSS] . It was achieved with the methods of combinatorial algebras, Groebner bases and Lyndon words, tools the effectiveness of which ultimately relies on the Diamond Lemma [B]. We are reminded of Bergman's remark, that some problems become more tractable when generalized to an ultimate degree. It may be that some of the struggle associated with super Lie algebras can be reduced by generalization. From the point of view of quantum groups, Lie algebras, superalgebras and Kac-Moody algebras are very singular limits. This paper suggests that the difficulties can be mitigated by working with quantum groups and passing to the interesting limits afterwards.

We deal with a class of algebras that forms a natural generalization of the quantum groups first defined systematically by Drinfel'd [D]. Simplicity is gained by a method

first suggested by Lusztig [L]. In this formulation the Cartan subalgebra disappears and there remains only the subalgebra generated by the positive simple root vectors. This structure also appears in the work of Kashiwara [Ks] and others in connection with crystal bases. The trick of eliminating the Cartan subalgebra from the scene was rediscovered by the author in his work on universal R-matrices [F]. It led to the study of "free differential algebras", for which a better term might be $q$-algebras.

One begins with the free, unital $\mathbf{C}$-algebra $\mathcal{B}$ on generators $\{e_i\}_{i\in\mathcal{N}=\{1,...,N\}}$ that should be regarded as positive Serre generators. The negative Serre generators are represented (replaced) by q-differential operators. The *parameters* are the values of a function $q:\mathcal{N}\times\mathcal{N}\to\mathbf{C},\ (i,j)\mapsto q_{ij}\neq 0$. They appear in the action of a set $\{\partial_i\}_{i\in\mathcal{N}}$ of q-differential operators on $\mathcal{B}$, this action being defined by $\partial_i x=0$ for $x\in\mathbf{C}\subset\mathcal{B}$ and iteratively for all $x\in\mathcal{B}$

$$\partial_i(e_jx)=\delta_i^jx+q_{ij}e_j\partial_ix\,. \tag{1.1}$$

Henceforth $\mathcal{B}$ will stand for the algebra $\mathcal{B}$ endowed with this $q$-differential structure. It is naturally graded, $\mathcal{B}=\bigoplus_{n=0,1,...}\mathcal{B}_n$, with $\mathcal{B}_0=\mathbf{C}$. A *constant* is an element $C\in\bigoplus_{n=1,2,...}\mathcal{B}_n$ with the property $\partial_iC=0,\ i=1,...,N$. For parameters $\{q_{ij}\}$ in general position there are no constants, constants exist only for values of the parameters that by that virtue will be called *singular*. The constants generate an ideal $I_q\subset\mathcal{B}$. The algebras of interest (quantum groups, quantized Kac-Moody algebras among them, see below) are the quotient algebras $\mathcal{B}'=\mathcal{B}/\mathcal{I}_q$, with the inherited q-differential structure. Here are some examples. In each case the parameters are supposed to be generic except for the explicitly imposed constraints.

(1) If for all $i\in\mathcal{N}$ we have $q_{ii}=-1$, then $I_q$ is generated by $\{e_i^2\}_{i\in\mathcal{N}}$. (2) If for all $(i,j),\ i\neq j$ we have $q_{ij}q_{ji}=1$, then $\mathcal{B}'$ is Manin's quantum plane. (3) If for all $(i,j),\ i\neq j$ there is a non-negative integer $k_{ij}$ such that

$$(q_{ii})^{k_{ij}}q_{ij}q_{ji}=1, \tag{1.2}$$

then $\mathcal{B}'$ is a quantized, generalized Kac-Moody algebra with Cartan matrix $A_{ij}=-k_{ij}$ for $i\neq j$. This last category includes quantized versions of the basic, generalized super-Kac-Moody algebras; that is, those that are based on a Cartan matrix. In fact, the distinction between Lie algebras and super Lie algebras disappears upon quantization (in the sense of Drinfel'd) and this may be taken as an indication that certain aspects of the theory actually become simpler after quantization. (4) A final example [F],[FG], to show that this category of algebras encompasses new and interesting possibilities, is the case that the parameters satisfy the single constraint $\prod q_{ij}=1$, where the product runs over all imbeddings of $\{i,j\}$ into $\{1,...,n\}$. The ideal is generated by a polynomial in $N$ variables. Examples of this kind play a role in quantized super Lie algebras* [FSS].

The problem of classification that we are working on here is a very natural one:

> *Find all the singular points in parameter*
> *space and describe the associated ideals in* $\mathcal{B}$.

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* Thanks to Cyrill Oseledetz for pointing this out.

Of course we shall not accomplish this. But we shall complete the following program. First we remark that the space of constants is generated by homogeneous constants. Let $Q$ be a set of natural numbers, with repetition, with cardinality $n$, and let $\mathcal{B}_Q$ be the subspace of $\mathcal{B}$ generated by the set of monomials $\{e_{i_1}...e_{i_n}\}$ where the indices run over all permutations of $Q$. Program:

*Under the stipulation that there are no constants in $\mathcal{B}_{Q'}$,*
*$Q'$ any proper subset of $Q$, find all the singular points,*
*and all the associated constants, in $\mathcal{B}_Q$.*

This problem had already been solved for the case $Q = \{1, 2, ..., n\}$. The complications introduced by repetitions in $Q$ will be recognized as similar to those that arise when one passes from Lie algebras to super Lie algebras and color super Lie algebras [MZ], and the methods of analysis will overlap. We find that the introduction of a Groebner basis in terms of generalized Lyndon words is useful, but another basis has been more effective, so far. The answer is given in terms of certain idempotents in the group algebra of $S_n$, and the lengths of certain orbits of cyclic subgroups acting in $S_nQ$.

*Relation to other work.*

The problematics of this paper is of course related to the vast litterature on quantum groups, especially to references already quoted. However, it is essentially different in that it explores a larger category. (We use the word in a non-technical sense.) It is sufficiently larger than the usual one of quantum groups to be interesting. There is the hope that some results may be easier to come by in this general setting. To my knowledge, this point of view is found in the following papers. First, there is the work of Varchenko [V] on the arrangement of hyperplanes. His emphasis is on quantum groups, but the general situation is clearly envisaged. The paper [R] explicitly calls for an exploration of generalizations of quantum groups, in precisely the same direction. But the work of Kharchenko [Kh] appears to be closest in scope to our work. This is perhaps not obvious at first sight, so it is necessary to sketch the Hopf algebra aspects of the algebras that are under investigation.

Consider the automorphisms $\{\mathcal{K}_i, K^i\}_{i=1,...,N}$ defined by $\mathcal{K}_i : e_j \mapsto (q_{ji})^{-1} e_j$, $\mathcal{K}^i : e_j \mapsto q_{ij} e_j$. Enlarge the algebra $\mathcal{B}$ by adjoining invertible elements $K_i, K^i$ that implement these automorphisms, and consider the algebra generated by $e_i, K_i, K^i$ with relations

$$K_i e_j (K_i)^{-1} = (q_{ij})^{-1} e_j.$$

This becomes a bialgebra with coproduct and antipode defined by

$$\Delta e_i = e_i \otimes 1 + K_i \otimes e_i, \quad \Delta K_i = K_i \otimes K_i, \quad i = 1, ..., N.$$

$$S(K_i) = (K_i)^{-1}, \; S(e_i) = -K_i e_i.$$

Kharchenko studies certain "quantum relations" in this algebra; they are precisely the constants. In the paper [Kh], is a result on the classification of these relations that coincides with a result in [FG]. The relationship to Universal R-matrices and Kac-Moody algebras was summarized in [FG], but only the paper [F] goes into detail.

*Summary and results.*
There is an old, partial result. Its scope limited in two ways:

(1) It is assumed that the parameters are such that there are no constants in $\mathcal{B}_{Q_j}$, $Q_j$ the subset of $Q$ obtained from $Q$ by reducing by 1 the incidence of $j$ in $Q$. In other words, one looks for new phenomena that first appear at polynomial order $n$ in the "most generic" case.

(2) The degrees of homogeneity are restricted to 0 and 1. That is, one considers, for some natural number $n$, the subspace $\mathcal{B}_{\underline{n}}$ , $\underline{n} = 12...n$ , of homogeneous polynomials of degree one in each generator $e_1, ..., e_n$.

Within the limitation of both stipulations, it was possible to get complete results [FG]. (See also [V] and [Kh].) The space of constants in $\mathcal{B}_{\underline{n}}$ is empty unless

$$\prod q_{ij} = 1, \tag{1.3}$$

and then the dimension of the space of constants is $(n-2)!$. An algorithm was given for the construction of all the constants in this case.

In this paper only the second of the two stipulations will be removed. We consider polynomials spanned by $e_{i_1}...e_{i_n}$, with fixed degrees of homogenity, $\underline{i} = \{i_1...i_n\}$ running over the permutations $\hat{Q}$ of an arbitrary set $Q = 1^{n_1}2^{n_2}...$, $\sum n_i = n$. To state the result we must consider the natural action of $S_n$ in $\hat{Q}$ and in $\mathcal{B}_Q$, as well as the action of $S_{n-1}$ in $\hat{Q}_j$ and in $\mathcal{B}_{Q_j}$. In the case $Q = \underline{n}$ , the product (1.3) runs over all pairs $(i,j) \subset \underline{n}$ , in general (1.3) is interpreted as a cocycle condition and the product runs over orbits of the cyclic subgroups $S_n^c$ and $S_{n-1}^c$. Complications arise whenever some of these orbits are short.

For a choice of the parameters $q_{ij}$, let $\chi,\ \chi_j$ be the number of orbits of $S_n^c$ in $\hat{Q}$, resp. $S_{n-1}^c$ in $\hat{Q}_j$, for which the cocycle condition (1.3) is verified, then *the dimension of the space of constants is* $\sum \chi_j - \chi$. Define certain iterated $q$-commutators $X^{\underline{i}}$, $\underline{i} = i_1...i_n$ , see Section 2.1.3, with the property that

$$\partial_j X^{\underline{i}} = 0,\ j \neq i_1,$$

and let $\hat{X}^j = \{X^{\underline{i}};\ i_1 = j\}$. *The subspace spanned by any one of these sets of polynomials contains all the constants.* This gives a practical method for the computation of the constants.

The result is both complete and of a comforting simplicity, but the demonstration is not satisfying, for we were unable to base the proof on a coherent strategy. Neither the old result, for the case $Q = \underline{n}$, nor the method used to obtain it, was brought to bear on the more general problem, because that method was lacking in elegance and probably not suited for more complicated situations. We first examine a subspace spanned by simple commutators of the form $[e_{i_1}...e_{i_{n-1}}, e_{i_n}]_{a(\underline{i})}$, with the commutation factor $a$ satisfying a certain cocycle condition. This space is certainly fundamental; it is the same as that spanned by the iterated commutators $X^{\underline{i}}$ and it was proved that it contains all the constants. The relation to Kac-Moody algebras and super Kac-Moody algebras is evident and it is felt that these q-commutators are easier to deal with than

the (anti-) commutators that arise in the limit when the commutation factors tend to $\pm 1$. This belief is based on the observation that those limiting cases are the most singular points in parameter space. To get this far we also had to find a trick that relates the constants in $\mathcal{B}_Q$ to the constants in $\mathcal{B}_{\underline{n}}$. A third method had to be used to finish the job. A basis built on Lyndon words, and an identification of Lyndon words with iterated commutators, of the type used in connection with superalgebras [MZ], gave us an existence lemma that was required to reach our goal. But that is the only effective use that we were able to make of this basis.

The first stipulation was essential. It is unclear which, if any, of the several methods used here will turn out to be of use in the future, when we shall try to eliminate it.

## 2. Constants and commutators.

For any ordered set $n_1, ..., n_k$ of positive integers set

$$n := \sum_{i=1}^{k} n_i,$$

and let $Q$ be the ordered set,

$$Q = \{1, ..., 1, 2, ..., 2, ..., k, ..., k\} = \{(1)_{n_1}, (2)_{n_2}, ..., (k)_{n_k}\}.$$

Let $\mathcal{B}_Q$ be the linear space spanned by the polynomials $e_{i_1}, ..., e_{i_n}$, where $i_1, ..., i_n$ runs over the set $\hat{Q} := S_n Q$ of distinct permutations of $Q$. This paper studies the constants in $\mathcal{B}_Q$ in the case that there are no constants in $\mathcal{B}_{Q'}$, for any proper subset $Q' \subset Q$.

*2.1. Simple and iterated commutators..*

*2.1.1. Simple commutators and cyclic permutations.* We write $\underline{i}$ for $i_1 ... i_n$, and sometimes for other index sets, when the context makes it clear which set is meant. Fix a function $a : \hat{Q} \to \mathbf{C} - \{0\}$, $\underline{i} \mapsto a(\underline{i}) \neq 0$ and let $\hat{A}_Q$ be the set of simple commutators

$$\hat{A}_Q := \{[e_{i_1} ... e_{i_{n-1}}, e_{i_n}]_{a(\underline{i})};\ \underline{i} \in \hat{Q}\}. \tag{2.1}$$

Let $S_n^c$ denote the cyclic subgroup of $S_n$, and $[sQ]^c \subset \hat{Q}$ the orbit of $S_n^c$ through $sQ \in \hat{Q}$. The function $a$ may be called a *commutation factor* ; compare [MZ].

*2.1.2. Proposition.* For $\{a(\underline{i})\}$ in general position, the set $\hat{A}_Q$ spans $\mathcal{B}_Q$. The full set of linear relations in $\mathbf{C}\hat{A}_Q$ is generated by relations of the following type. For each $sQ \in \hat{Q}$, such that

$$\prod_{\underline{i} \in [sQ]^c} a(\underline{i}) = 1\,; \tag{2.2}$$

that is, such that $a(\underline{i}), [sQ]^c$ defines an $S_n^c$ cocycle, there is a unique relation

$$\sum_{\underline{i} \in [sQ]^c} C(\underline{i})[e_{i_1} ... e_{i_{n-1}}, e_{i_n}]_{a(\underline{i})} = 0\,, \tag{2.3}$$

with complex coefficients $C(\underline{i})$, $\underline{i} \in [sQ]^c$ all different from zero.

More precisely, this relation has the form

$$\sum_{\alpha=0,1,...} \left( \prod_{\beta<\alpha} a(\tau^\beta \underline{i}) \right) \tau^\alpha [e_{i_1}...e_{i_{n-1}}, e_{i_n}]_{a(\underline{i})} = 0. \tag{2.4}$$

where $\underline{i} = sQ$ $(s \in S_n)$ is fixed, $\tau$ is a generator of $S_n^c$ and the sum extends to the orbit of $S_n^c$ through $sQ$. The operator

$$P(\underline{i}) := \sum_{\alpha=0,1,...} \left( \prod_{\beta=0,...,\alpha-1} a(\tau^\beta \underline{i}) \right) \tau^\alpha \tag{2.5}$$

is thus, up to a numerical factor, an idempotent in the group algebra of $S_n$. When $\hat{A}_Q$ fails to span $\mathcal{B}_Q$, a supplemental basis is given by any set of monomials that contains exactly one element from each orbit of $S_n^c$ for which the cocycle condition holds. More generally, the *orbital polynomial*

$$u = \sum_\alpha C_\alpha \tau^\alpha e_{i_1}...e_{i_n} \tag{2.6}$$

is in $\mathbf{C}\hat{A}_Q$ if and only if it satisfies the *dual cocycle condition*

$$\sum_\alpha \left( \prod_{\beta \geq \alpha} a(\tau^\beta \underline{i}) \right) C_\alpha = 0, \tag{2.7}$$

and a supplemental basis is made up of any collection of orbital polynomials, one for each orbit, provided that they all violate the dual cocycle condition.

*2.1.3. Iterated commutators.* For any permutation $sQ = \{i_1...i_n\}$ of $Q$, $s \in S_n$, and commutation factors $a(i_1...i_p) \neq 0,\ p = 1, ..., n$, let

$$X^{i_1} = e_{i_1}, \quad X^{i_1...i_p} = [X^{i_1...i_{p-1}}, e_p]_{a(i_1...i_p)}, \quad p = 1, ..., n.$$

*2.1.4. Proposition.* For commutation factors $a(i_1...i_p)$ in general position the set

$$\hat{X} := \{X^{i_1...i_n}\ ;\ i_1...i_n \in \hat{Q}\} \tag{2.8}$$

spans $\mathcal{B}_Q$.

*Proof.* By induction; in fact, in this case $\mathbf{C}\hat{X} = \mathbf{C}\hat{A}_Q$.

*2.2. Constants.*

*2.2.1.* From now on the commutation factors will be chosen as follows. Fix a function $q : \hat{Q} \otimes \hat{Q} \to \mathbf{C} - \{0\},\ (i,j) \mapsto q_{ij} \neq 0$ and set

$$a(i_1...i_p) = q_{i_p i_1} ... q_{i_p i_{p-1}}. \tag{2.9}$$

The complex numbers $q := \{q_{ij}\}$ are *the parameters.** The action of the $q$-differential operators $\partial_i$ in $\mathcal{B}$ was defined in Section 1. Let $u = e_{i_1}...e_{i_{p-1}}$, then for all $x \in \mathcal{B}$,

$$\partial_{i_p}(ux) = (\partial_{i_p} u)x + a(u, i_p) u \partial_{i_p} x, \quad a(u, i_p) := a(i_1...i_p),$$

and if $p = n$, then $[u, e_{i_n}]_{a(u,i_n)} \in \hat{A}_Q$. The mapping defined by $u \mapsto \{a(u,i)\}_{i=1,...,N}$ is a grading of $\mathcal{B}$; graded elements are said to be *homogeneous.*

The action of $e_i$ on $\mathcal{B}$ and on the quotient $\mathcal{B}' = \mathcal{B}/\mathcal{I}_q$ defined by $u \mapsto [u, e_i]_{a(u,i)}$ generalizes the adjoint action of quantum groups. (A proof of this statement will be published.)

The significance of the choice (2.9) in the present context is that

$$\partial_j [u, e_i]_{a(u,i)} = [\partial_j u, e_i]_{a(u,i) q_{ji}}, \tag{2.10}$$

for all pairs $(i,j)$ including the case that $i = j$, and that, consequently,

$$\partial_j X^{i_1 ... i_p} = 0,\ j \neq i_1. \tag{2.11}$$

The cocycle condition (2.2) reduces, in the case of long orbits (orbits of length $n$) to

$$\prod q_{ij} = 1, \tag{2.12}$$

where the product runs over all $n(n-1)$ pairs $(i,j) \subset \hat{Q}$. Example: $Q = 1123$, Eq.(2.12) reads $(q_{11} q_{12} q_{21} q_{13} q_{31})^2 q_{23} q_{32} = 1$.

*2.2.2.* Let $Q_i, i = 1, ..., k$ be the sets obtained from $Q$ by removing one element, thus reducing $n_i$ by one, $i = 1, ..., k$, respectively, $\hat{Q}_j$ the set of permutations of $Q_j$ and $\mathcal{B}_{Q_j}, j = 1, ..., k$ the corresponding spaces of polynomials.

*2.2.3. Stipulation* . It is supposed from now on, throughout Sections 2 and 3, that the parameters $q_{ij}$ are such that, for each $j = 1, ..., k$ separately, (a) $\hat{A}_{Q_j}$ spans $\mathcal{B}_{Q_j}$ and (b) there are no constants in $\mathcal{B}_{Q_j}$ ; for each $j = 1, ..., k$.

The case $Q = \{1^n\}$ shows that (a) and (b) are independent. But it will turn out that, if $Q \neq 1^n$, then (b) implies (a). See corollary 2.2.6.

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* For another interpretation see 3.2.6.

**2.2.4. Theorem.** If $C \in \mathcal{B}_Q$ is a constant: $\partial_j C = 0$ for $j = 1, ..., k$, then $C \in \mathbf{C}\hat{A}_Q$.

*Proof.* Let $\{\mathcal{O}_\alpha\}$ be the collection of orbits of $S_n^c$ in $\hat{Q}$ or in $\mathcal{B}_Q$ on which the cocycle condition (2.2) holds. In view of Eq.(2.9), this set is non-empty iff and only if

$$\prod q_{i_1 i_2} = 1. \tag{2.13}$$

We refer to them as the *singular* orbits. Select a monomial $u(\alpha) \neq 0$ from each singular orbit $\mathcal{O}_\alpha$ of $S_n^c$ in $\mathcal{B}_Q$. A monomial cannot satisfy the cocycle condition, therefore

$$\mathcal{B}_Q = \bigoplus_\alpha \mathbf{C} u(\alpha) \ \oplus \ \mathbf{C}\hat{A}_Q. \tag{2.14}$$

A constant in $\mathcal{B}_Q$ is a polynomial

$$C = \sum_\alpha C(\alpha)\, u(\alpha) + \sum_{\underline{i} \in \hat{Q}} C'(\underline{i})[e_{i_1} ... e_{i_{n-1}}, e_{i_n}]_{a(\underline{i})}, \tag{2.15}$$

such that, for $j = 1, ..., k$,

$$\partial_j C = \sum_\alpha C(\alpha)\, \partial_j u(\alpha) + \sum_{\underline{i} \in \hat{Q}} C'(\underline{i})[\partial_j(e_{i_1} ... e_{i_{n-1}}), e_{i_n}]_{a(\underline{i}) q_{j i_n}} = 0. \tag{2.16}$$

Eq.(2.10) was used. The existence of a nontrivial solution of (2.16) implies the existence of a nontrivial solution of the following equation,

$$\sum_\alpha C(\alpha)\, \partial_j u(\alpha) + \sum_{\underline{i} \in \hat{Q}_j} C_j'(\underline{i})[\, e_{i_2} ... e_{i_{n-1}}, e_{i_n}]_{b_j(\underline{i})} = 0. \tag{2.17}$$

Here

$$b_j(\underline{i}) = a(j\underline{i}) q_{j i_n}, \tag{2.18}$$

and we notice that

$$\prod_{s \in S_{n-1}^c} b_j(s\underline{i}) = \prod_{s \in S_n^c} a(sj\underline{i}) = \prod q_{i_1 i_2}, \ j = 1, ..., k, \tag{2.19}$$

where one or the other (but not both unless $Q = 1^n$) of the first two products may run more than once over an orbit of $S_{n-1}^c$, resp. $S_n^c$. The third product is over all imbeddings of $(i_1, i_2)$ into $Q$. We conclude that, if the cocycle condition (2.13) holds, then in each $\mathcal{B}_{Q_j}$, with commutation factors $b_j(\underline{i})$, all the long orbits are singular, and that if (2.13) does not hold, then none of the orbits in $\mathcal{B}_{Q_1}, ..., \mathcal{B}_{Q_k}$ are singular. In this latter case the first part of (2.17) is absent.

We wish to show that Eq.(2.16) requires that $C(\alpha) = 0$. Consider first the case that $Q = \underline{n} := \{1, ..., n\}$; that is, the case that there is no repetition in $Q$, each index

occurring just once. Then all the orbits are long and the collection $\{u(\alpha)\} = \{u(\underline{i})\} = \{e_{i_1}...e_{i_{n-1}}e_1\}$ contains exactly one monomial for each orbit of $S_n^c$ in $\hat{Q}$. Hence

$$\sum C(\alpha)u(\alpha) = ve_1, \quad v \in \mathcal{B}_{Q_1},$$

and for $j \neq 1$,

$$\sum C(\alpha)\partial_j u(\alpha) = (\partial_j v)e_1$$

contains at most one monomial from each orbit of $S_{n-1}^c$. Since all these orbits are singular, the first term in (2.17) plays exactly the same role as the first term in (2.15), complementing $\mathbf{C}\hat{A}_{Q_1}$ in $\mathcal{B}_{Q_1}$. In other words, the two sums in (2.17) must vanish separately, $\partial_j v = 0$ for all $j$, and hence by the stipulation, $v = 0$. The theorem is proved for this case.

*2.2.5. Lemma.* Let $\underline{n} = \{1, 2, ..., n\}$ and let $\phi : \underline{n} \to Q, i' \mapsto i$ be the natural mapping that takes e.g. 1234 to 1122. Let $\{q'_{i'j'} = q_{ij}\}$ be the parameters in $\mathcal{B}_{\underline{n}}$. Let $C$ be a constant in $\mathcal{B}_{\underline{n}}$, then a constant in $\mathcal{B}_Q$ is obtained by replacing $e_{i'} \mapsto e_i, i' = 1, ..., n$. Conversely, every constant in $\mathcal{B}_Q$ can be obtained in this way.

*Proof.* The direct statement is clear. To prove the converse, let $C$ be any homogeneous constant in $\mathcal{B}_Q$. In $C$, replace each generator $e_i, i = 1, ..., k$, by the sum $\sum_{i'} e_{i'}$, where the sum runs over all $i' \in \phi^{-1}(i)$. The resulting polynomial $C'$ is a constant. Furthermore, each homogeneous component of $C'$ is a constant, and one of these components, $C'_1$, say, is in $\mathcal{B}_{\underline{n}}$. And now $C$ is recovered from $C'_1$ by replacing $e_{i'} \mapsto e_i, i' = 1, ..., n$.

The proof of the theorem is completed by noting that $C'_1$ is certainly in $\mathbf{C}\hat{A}_{\underline{n}}$, since there is no repetition in $\underline{n}$. Expressing $C'_1$ as a sum of commutators (with the requisite commutation factors determined by the parameters), we finally replace $e_{i'}$ by $e_i$ to obtain an expression for $C$ as an element of $\mathbf{C}\hat{A}_Q$. The theorem is proved.

*2.2.6. Corollary.* Assume that $\mathbf{C}\hat{A}_Q$ fails to span $\mathcal{B}_Q$. Then the cocycle condition (2.13) holds and (2.19) shows that the long orbits (if any) in $\hat{Q}_1, ...\hat{Q}_k$ are singular. Then usually there are constants in $\mathcal{B}_Q$, the only exception being the case $Q = 1^n$, when all the orbits of $S_{n-1}$ are short, and $q_{11} = 1$. Hence part (a) of the stipulation is redundant if $Q \neq 1^n$.

## 3. Constants and iterated commutators.

See definitions in 2.1.3, and let

$$\hat{X}^j := \{X^{ji_2...i_n}, i_2...i_n \in \hat{Q}_j\}, \quad j = 1, ..., k \ , \tag{3.1}$$

For parameters in general position we have $\mathcal{B}_Q = \mathbf{C}\hat{X} = \cup_{j=1,...,k}\mathbf{C}\hat{X}^j$, a disjoint union. In general we shall show that, not only are the constants in $\mathbf{C}\hat{X}$; they are in fact all contained in each and every one of the subspaces $\mathbf{C}\hat{X}^j, \ j = 1, ..., k$.

*3.1. Constants in $\mathbf{C}\hat{X}^j$.*

Here we shall first construct an algorithm for the determination of constants, without at first inquiring about its effectiveness.

*3.1.1.* Define $Y_j^{i_2...i_n}$ by

$$\partial_j X^{i_1...i_n} = \delta_j^{i_1}(1 - q_{i_1 i_2} q_{i_2 i_1}) Y_j^{i_2...i_n},$$

then

$$Y_j^{i_2} = e_{i_2},\ Y_j^{i_2...i_p} = [Y_j^{i_2...i_{p-1}}, e_{i_p}]_{b_j(i_2...i_p)},$$

with commutation factors

$$b_j(i_2...i_p) = q_{j i_p} a(j i_2 ... i_p),$$

already introduced in (2.18) for $p = n$.

Let $\hat{Y}_j := \{Y_j^{i_2...i_n};\ i_2...i_n = \hat{Q}_j\}$, where $\hat{Q}_j$ is the set of permutations of $Q_j$.

*3.1.2.* Fix $j \in \{1, ..., k\}$. A constant in $\mathbf{C}\hat{X}^j$ is an element $\sum_{\underline{i} \in \hat{Q}_j} C(\underline{i}) X^{j i_2 ... i_n}$ (where $\underline{i}$ is short for $i_2...i_n$) such that, for $n > 2$,*

$$\sum_{\underline{i} \in \hat{Q}_j} C(\underline{i})(1 - q_{j i_2} q_{i_2 j}) Y_j^{i_2...i_n} = \sum_{\underline{i} \in \hat{Q}_j} C(\underline{i})(1 - q_{j i_2} q_{i_2 j})[Y_j^{i_2...i_{n-1}}, e_{i_n}]_{b_j(\underline{i})} = 0. \quad (3.2)$$

Consequently, a constant in $\mathbf{C}\hat{X}^j$ implies that there is a relation among the elements of the set $\{\hat{Y}_j^{i_2...i_n},\ i_2...i_n \in \hat{Q}_j\}$. Conversely, if such a relation exists, then either there is a constant in $\mathbf{C}\hat{X}^j$, or else there is a relation among the elements of the set $\hat{X}^j$. By the stipulation, neither situation can arise if the set $Q$ is replaced by $Q_i$, any $i$, so we can conclude that, for each value of $i_n$, the set of $Y_j^{i_2...i_{n-1}}$'s that appear inside the commutator (3.2) is linearly independent. Therefore, the existence of a solution of (3.2) is equivalent to the existence of a solution of the following equation,

$$\sum_{\underline{i} \in \hat{Q}_j} C'(\underline{i})[e_{i_2}...e_{i_{n-1}}, e_{i_n}]_{b_j(\underline{i})} = 0. \quad (3.3)$$

By Proposition 2.1.2, this in turn depends on the cocycle condition for the commutation factors; namely, such relations imply that there is $s \in S_{n-1}$ such that

$$\prod_{\underline{i} \in [sQ_j]^c} b_j(\underline{i}) = 1, \quad (3.4)$$

where $[sQ_j]^c$ is the orbit through $sQ_j$ of the cyclic subgroup $S_{n-1}^c$ of the group $S_{n-1}$ of permutations of $i_2...i_n$. So we have, in all cases, the following result.

* Under the stipulation, when $n > 2$, the factor in parenthesis cannot vanish.

*3.1.3. Proposition.* For every orbit $[sQ_j]^c$ of $S^c_{n-1}$ for which the constraint (2.19) holds, there is a linear combination of the elements in $\hat{X}^j$ that is annihilated by $\partial_j$; and this subset of $\mathbf{C}\hat{X}^j$ spans the subspace of constants in $\mathbf{C}\hat{X}^j$.

*3.1.4. Remark. Examples.* This set of constants is not necessarily linearly independent; it may even be empty. The simplest example of this is $Q = 11$ and $q_{11} = 1$, which satisfies the cocycle condition $q_{11}^2 = 1$. A less trivial example is $Q = 112$, $Q_1 = 12$, with $q_{11}q_{12}q_{21} = -1$, a solution of the cocycle condition $(q_{11}q_{12}q_{21})^2 = 1$; if this is the only constraint then there are no constants in $\mathcal{B}_{112}$.

In the case of an orbit of (maximal) length $n$, we found - Eq.(2.19) - that the condition (3.4) reduces to

$$\prod_{\underline{i}\in[\tilde{Q}_j]^c} a(\underline{i}) = \prod q_{ij} = 1, \tag{3.5}$$

where $\underline{i} = i_1...i_n,\ \tilde{Q}_j = jsQ_j$ and the second product runs over all distinct imbeddings of $\{i,j\}$ in $Q$. Complications arise when one or the other of the orbits $[sQ_j]^c$ or $[\tilde{Q}_j]^c$ (but not both unless $Q = 1^n$) is short. In the first case the condition (3.5), though it is always necessary, may not be sufficient. In the second case the constant found by solving (3.3) may turn out to vanish identically. An example of this is the case $Q = 1122$; the cocycle condition $q_{11}q_{22}(q_{12}q_{21})^2 = 1$ on the short orbit through 1212 is stronger than the cocycle condition on the long orbit through 1122 and when it holds there are no constants. This will all be sorted out in 3.2.5.

The case when $Q = 12...n$, without repetitions, when none of the above complications can arise, was investigated elsewhere. The result has implications for the general case envisaged here. The stipulation 2.2.3 remains in force.

**3.1.5. Theorem.** [FG] When $Q = 12...n$, without repetitions, then there is no constant in $\mathcal{B}_Q$ unless the constraint (3.5) holds. In this case the space of constants in $\mathcal{B}_Q$ has dimension $(n-2)!$ and is a subspace of $\mathbf{C}\hat{X}^j$ for each $j = 1,...,k$.

*3.1.6.* In this case the co-dimension of $\mathbf{C}\hat{X}$ in $\mathcal{B}_Q$ is $(n-1)!$ (the number of distinct orbits of $S^c_n$ in $\hat{Q}$) and each of the $n$ sets $\hat{X}^j$ is linearly independent. The $(n-1)!$ relations among the elements of $\hat{X}$ relate the constants in $\mathbf{C}\hat{X}^i$ to the constants in $\mathbf{C}\hat{X}^j$.

In the general case we have an algorithm for constructing all the constants in $\mathbf{C}\hat{X}^j$, starting with Eq.(3.2), but it remains to be shown that every $\mathbf{C}\hat{X}^j$ contains all the constants in $\mathcal{B}_Q$. This can be shown by using Theorem 3.1.5 and Lemma 2.2.5, but we shall present a proof that does not rely on 3.1.5.

*3.2. Lyndon words.*

*3.2.1. Lyndon words and good words.* We shall introduce another basis for $\mathcal{B}_Q$. The starting point is the natural basis of distinct monomials $e_{i_1}...e_{i_n}$ that we shall now abbreviate as $i_1...i_n$. Let us refer to the generators $e_i = i$ as ‘letters’.

Step One. In any monomial $i_1...i_n$ introduce parentheses as follows. Starting at the left end, suppose the sequence descends (in the sense of the natural order) as far as (and including) $i_a$; open a parenthesis to the left of $i_a$. This $i_a$ may be immediately repeated, any number of times. Once past these repetitions, close the parenthesis just before the first occurrance of an $i_b \leq i_a$. Then continue towards the right, applying the same rules to the rest of the sequence. Here are some examples,

$$(1122),\ (12)(12),\ (122)1,\ 2(112),\ 2(12)1,\ (22)(11).$$

Step Two. A word is either a letter or a group of letters in a parenthesis of the type that was constructed. (That is, the sequence inside a word has the form $(i_a)^m i_r i_s...$ where $i_r > i_a, i_s > i_a, ....$) Order the words lexicographically. Then apply the procedure of Step One to the words. In the example all the words but one is already in descending order and only one new parenthesis appears,

$$(1222),\ ((12)(12)),\ (122)1,\ 2(12)1,\ (22)(11).$$

This process comes to an end after a finite number of steps. However, we shall not need to go beyond the first step. We refer to the sequences obtained after the first step as the good sequences, and to the associated words as good words. Good words contain no nested parentheses.

Next, we interpret each word as an iterated commutator:

$$(ij...) = (e_i e_j...) = X^{ij...}. \tag{3.6}$$

The set of all compound words that result from applying Step One to each of the permutations of any specific monomial form a basis for the corresponding linear space. In particular, the set of all good compound words associated with $Q$ is a basis for $\mathcal{B}_Q$.

*3.2.2.* Now we fix $j \in \{1, ..., k\}$ and notice that, in this basis, when the parameters are in general position, the subspace $M_j$ of all compound words annihilated by $\partial_i,\ i \neq j$, consists of all those words in which each and every parenthesis begins with $j$, and that the dimension of this subspace is the same as that of $\mathbf{C}\hat{X}^j$. In the special case, when the cocycle condition is satisfied on one or more orbits, the dimension of $M_j$ is stable, while that of $\mathbf{C}\hat{X}^j$ generally is not. Thus it is obvious that $M_j$ will contain all the constants, but it is *a priori* possible that $\mathbf{C}\hat{X}^j$ do not. It is important to establish that, in fact, it does. Towards this end we have so far established the following.

*3.2.3. Lemma.* For each $j \in \{1, ..., k\}$ there exists a fixed linear subspace $M_j$ of $\mathcal{B}_Q$ with the following properties. (a) For generic parameters; that is, for parameters $q_{ij}$ in general position, the set $\hat{X}^j$ is a basis for $M_j$. In the exceptional case when the cocycle condition is satisfied on some set of orbits (the "singular" orbits), $M_j$ contains all the constants.

**3.2.4. Theorem.** The space $\mathbf{C}\hat{X}^j$, for any $j \in \{1, ..., k\}$ contains the space of constants in $\mathcal{B}_Q$. Hence this space is precisely $\mathbf{C}\hat{X}^i \cup \mathbf{C}\hat{X}^j$, for any pair $(i, j),\ i \neq j$.

Note that the stipulation 2.2.3 is in force.

*Proof.* The spaces $M_j$ are disjoint and span $\mathcal{B}_Q$. Let $M_j'$ be a complement of $\mathbf{C}\hat{X}^j$ in $M_j$. Then the union of all the $M_j'$'s form a compliment to $\mathbf{C}\hat{X}$ in $\mathcal{B}_Q$. We know, from 2.2.4, that such a complement, no matter how it is chosen, cannot contain a constant. So $M_j'$ cannot contain a constant. Since $M_j$ contains all the constants, so does $\mathbf{C}\hat{X}^j$.

*3.2.5. The dimension of the space of constants.* Let $\chi$, resp. $\chi_j$, be the number of distinct orbits in $Q$, resp. $Q_j$ on which the cocycle condition holds, the singular orbits. Assume that there are no constants in $\mathcal{B}_{Q_j}, j = 1, ..., k$. Then the dimension $\#(Q)$ of the space of constants in $\mathcal{B}_Q$ is

$$\#(Q) = \sum_{j=1}^{k} \chi_j - \chi. \tag{3.7}$$

In the case that there are no short orbits, in $Q$ or in $Q_1, ..., Q_k$, these numbers are

$$\chi = \frac{|\hat{Q}|}{n} = \frac{(n-1)!}{n_1!...n_k!}, \quad \chi_i = \frac{n_i(n-2)!}{n_1!...n_k!}, \quad \#(Q) = \frac{(n-2)!}{n_1!...n_k!}. \tag{3.8}$$

Note that it is only in the case that $n_j = 1$ that the singular orbits in $\hat{Q}_j$ are in 1:1 correspondence with a basis for the space of constants.

An example. Let $Q = 1122$; there is one long orbit and one short orbit. On the long orbits the cocycle condition is

$$\prod q_{ij} = \big(q_{11}q_{22}(q_{12}q_{21})^2\big)^2 = 1.$$

If this is not satisfied, then there are no constants. If it holds, then $\chi_1 = \chi_2 = 1$. In $Q$ there is a long orbit, on which the cocycle condition is satisfied. But there is also a short orbit, for which the condition is $q_{11}q_{22}(q_{12}q_{21})^2 = 1$. If it holds, then $\chi = 2$ and $\#(Q) = 0$. If it does not hold; that is, if $q_{11}q_{22}(q_{12}q_{21})^2 = -1$, then $\chi = 1$ and $\#(Q) = 1$. This example has applications to superalgebras.

*3.2.6. A question of rigour.* Throughout this paper we have used language that is slightly abusive. On the one hand, we refer to an algebra $\mathcal{B}$ (with differential structure) that is defined only once a set of parameters is fixed. On the other hand, there are phrases like "when $q_{12}q_{21} = 1$" that suggests variation of the parameters. It would be more appropriate to define the algebra as one generated by $\{e_i\}$ and $\{q_{ij}\}$, with relations that make the latter commute with the former and among themselves. Rather than impose constraints on the parameters, one would consider a family of quotients algebras obtained by dividing by polynomial ideals (polynomials in the $q_{ij}$ in this instance). No concept of continuity with respect to variation of the parameters is invoked.

**4. Multiple constraints.**

Consider the case $Q = 123$. If there are no constants in the spaces $\mathcal{B}_{Q_j}$ associated with the subsets $Q_j$, then there is precisely one constant in $\mathcal{B}_Q$ if and only $\prod q_{ij} = 1$. But if $q_{12}q_{21} = 1$, then there is a constant in $\mathcal{B}_{Q_3}$, and in this case the cocycle condition for $\hat{Q}$ is unrelated to the dimension of the space of constants. This phenomenon is central to further study of the structure of $\mathcal{B}' = \mathcal{B}/\mathcal{I}_q$.

**Appendix. Examples.**

For $i \neq j$, let $\sigma_{ij} := q_{ij}q_{ji}$.

$\underline{Q = \{1^n\}.}$ All orbits are short, $a(1^n) = (q_{11})^{n-1}$, $b_1(1^{n-1}) = (q_{11})^n$. There is a singular orbit in $\hat{Q}$ if $q_{11}^{n-1} = 1$, then $\mathbf{C}\hat{A}_Q$ is empty and there is no constant. There is a constant $(e_1)^n$ iff $(q_{11})^n = 1$, $q_{11} \neq 1$; it is in $\mathbf{C}\hat{A}_Q$.

$\underline{Q = \{12\}.}$ All orbits are long; $a(12)a(21) = q_{12}q_{21} =: \sigma_{12}$. Singular case is $\sigma_{12} = 1$, then the constant is $[e_1, e_2]_{q_{21}} \in \hat{A}_Q$. The constant $[e_1, e_2]_{q_{21}} = X^{12}$ is proportional to the constant $[e_2, e_1]_{q_{12}} = X^{21}$.

$\underline{Q = \{1^{m+1}2\}.}$ The unique orbit in $\hat{Q}$ is long; cocycle condition $\big((q_{11})^m\sigma_{12})\big)^{m+1} = 1$. The orbit in $Q_1$ is long and thus singular. The orbit in $Q_2$ is short, $b_2(1...1) = (q_{11})^m\sigma_{12}$.

(a) $(q_{11})^m\sigma_{12} = 1$. All orbits are singular; one constant, the Serre relation $X^{21...1}$. $(\chi = \chi_1 = \chi_2 = 1)$

(b) $q_{11}\sigma_{12} \neq 1$. The short orbit is not singular; no constant. $(\chi = \chi_1 = 1, \chi_2 = 0)$

$\underline{Q = \{123\}.}$ All orbits are long, and singular when $\sigma_{123} := \sigma_{12}\sigma_{23}\sigma_{31'} = 1$. One constant. $(\chi = 2, \chi_1 = \chi_2 = \chi_3 = 1)$ The constant was found in [F].

$\underline{Q = \{1122.\}.}$ There are two orbits in $\hat{Q}$. If $q_{11}q_{22}\sigma^2 = 1$, they are both singular; unique orbits in $\hat{Q}_1, \hat{Q}_2$ are also singular, no constants. $(\chi_1 = \chi_2 = 1, \chi = 2)$ When $q_{11}q_{22}\sigma^2 = -1$ the short orbit in $\hat{Q}$ is nonsingular; one constant. $(\chi = \chi_1 = \chi_2 = 1)$

$\underline{Q = \{1123.\}.}$ All orbits are long, singular; 1 constant. $(\chi = 3,\ \chi_1 = 2,\ \chi_2 = \chi_3 = 1)$.

$\underline{Q = \{1^{2m+1}22.\}.}$ There are $m+1$ orbits in $\hat{Q}$, all of them long and thus singular when $(q_{11})^{2m(2m+1)}(q_{22})^2\sigma_{12}^{2(2m+1)} = 1$. There are $m+1$ orbits in $\hat{Q}_1$, one long orbit in $\hat{Q}_2$, so if they are all singular there is one constant. But one of the orbits in $\hat{Q}_1$ is short, and if $(q_{11})^{m(2m+1)}q_{22}\sigma_{12}^{2m+1} = -1$, then it is not singular and then there is no constant.

$\underline{Q = \{1^{2m}22.\}.}$ There are $m+1$ orbits in $\hat{Q}$, one of them short. There are $m$ orbits in $\hat{Q}_1$ and 1 in $\hat{Q}_2$, all long. So there is one constant if the short orbit in $\hat{Q}$ is nonsingular; that is, if $(q_{11})^{m(2m-1)}q_{22}\sigma_{12}^{2m} = -1$, otherwise none.

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